SOME REMARKS ON NAGUMO'S THEOREM

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Abstract. We provide a simpler proof for a recent generalization of Nagumo's uniqueness theorem and we show that not only is the solution unique but the successive approximations converge to the unique solution.

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1. Introduction

Nagumo's remarkable theorem [7] for the Cauchy problem

$$x'(t) = f(t, x(t)) \tag{1.1}$$

with initial data

$$x(0) = 0, \tag{1.2}$$

where $a > 0$ and $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, states that (1.1)-(1.2) has a unique solution if

$$|f(t, x) - f(t, y)| \leq \frac{|x - y|}{t} \tag{1.3}$$

for $t \in (0, a]$ and x, $y \in \mathbb{R}^n$ with $|x|$, $|y| \leq M$ for some $M > 0$. This result improves considerably the classical Lipschitz condition. Among the various generalizations that appeared in the research literature, the most far-reaching was recently obtained in [4]. It states that uniqueness holds if $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, with

$$\frac{f(t, x)}{u'(t)} \to 0 \tag{1.4}$$

as $t \downarrow 0$, uniformly in $|x| \leq M$ for some $M > 0$, and satisfies

$$|f(t, x) - f(t, y)| \leq \frac{u'(t)}{u(t)}\, \omega(|x - y|), \tag{1.5}$$

for $t \in (0, a]$ and x, $y \in \mathbb{R}^n$ with $|x|$, $|y| \leq M$, where u is an absolutely continuous function on $[0, a]$ with $u(0) = 0$ and $u'(t) > 0$ a.e. on $[0, a]$, and where ω belongs to the class $\mathcal{F}$ of strictly increasing functions $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ and such that

$$\int_0^r \frac{\omega(s)}{s}\, ds \leq r, \qquad r > 0. \tag{1.6}$$

Notice that any strictly increasing continuous function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(s) \leq s$ for $s \geq 0$ belongs to the class $\mathcal{F}$. There are also functions $\omega \in \mathcal{F}$ for which $\omega(r_n) > r_n$ for all $n \geq 1$, along an appropriate sequence $r_n \downarrow 0$ cf. [4].

The object of this note is to give a simpler proof of this uniqueness result and to show that the hypotheses ensure not only uniqueness but also the convergence of the successive approximations. For this we adapt to the present context an approach that was developed in [1] to deal with the classical Nagumo theorem.

2. Alternative proof of uniqueness

The aim of this section is to provide a simpler proof of the uniqueness result in [4]. For this, we first derive a useful property of functions in the class $\mathcal{F}$.

Lemma 2.1. *If $\omega \in \mathcal{F}$ then $\omega(s) \le e\,s$ for $s \ge 0$.*

Proof. For $s > 0$ we have

$$\begin{aligned} s &\ge \int_0^s \frac{\omega(r)}{r}\,dr \ge \int_{s/e}^s \frac{\omega(r)}{r}\,dr \\ &\ge \omega\Big(\frac{s}{e}\Big)\int_{s/e}^s \frac{1}{r}\,dr = \omega\Big(\frac{s}{e}\Big) \end{aligned}$$

which yields the statement. □

Remark 2.2. The previous result might seem to indicate that we should simply set $\omega(s) = e\,s$ in (1.5) and dispense altogether with the class $\mathcal{F}$. However, in Nagumo's classical theorem (with $u(t) = t$ and $\omega(s) = s$) the growth of the coefficient $\frac{1}{t}$ as $t \downarrow 0$ is optimal: for any $\alpha > 1$ there exist continuous functions f satisfying (1.3) with the right-hand side multiplied by α but for which (1.1)-(1.2) has nontrivial solutions [1]. Thus replacing $\omega(s)$ by $s \mapsto e\,s$ is not an option.

A key role in our approach is the following Gronwall-type integral inequality (see [2, 5] for the classical Gronwall inequality and [3, 6] for generalizations in directions different to ours).

Lemma 2.3. *Let $u : [0, a] \to \mathbb{R}$ be absolutely continuous, nondecreasing and such that $u(t) > 0$ for $t > 0$. If $v : [0, a] \to \mathbb{R}$ is continuous, nonnegative, such that $v(t) = o(u(t))$ as $t \to 0^+$, and*

$$v(t) \le \int_0^t \frac{\omega(v(s))}{u(s)}\,u'(s)\,ds, \quad 0 < t \le a,$$

for some $\omega \in \mathcal{F}$, then v must be identically zero.

Proof. From Lemma 2.1 it follows that the integral is well-defined. Assume v is not the zero function. From $\frac{v(t)}{u(t)} \to 0$ as $t \to 0^+$ it follows that there exists $0 < \delta \le a$ such that $v(t) \le u(t)$ for $0 < t \le \delta$. Let

$$\varepsilon = \frac{v(t_0)}{u(t_0)} = \sup_{0<t\le\delta} \left\{\frac{v(t)}{u(t)}\right\} > 0$$

with $t_0 \in (0, \delta]$. We deduce that

$$\begin{aligned} \varepsilon u(t_0) = v(t_0) &\le \int_0^{t_0} \omega(v(s))\,\frac{u'(s)}{u(s)}\,ds \\ &< \int_0^{t_0} \omega(\varepsilon u(s))\,\frac{u'(s)}{u(s)}\,ds \\ &= \int_{\varepsilon\,u(0)}^{\varepsilon u(t_0)} \frac{\omega(r)}{r}\,dr \\ &\le \int_0^{\varepsilon u(t_0)} \frac{\omega(r)}{r}\,dr \le \varepsilon u(t_0), \end{aligned}$$

which is a contradiction. Thus v is identically zero. □

This enables us to give a simple proof of the main result of [4]:

Theorem 2.4. *If f is continuous and satisfies (1.4) and (1.5), then (1.1)-(1.2) has a unique solution.*

Proof. The local existence of a solution is guaranteed by Peano's theorem [5]. As for uniqueness, let $x(t), y(t)$ be two solutions of (1.1)-(1.2) for $0 < t \leq a$. In view of (1.4), given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(s,x)| \leq \varepsilon u'(s)$ for $0 < s \leq \delta$ and $|x| \leq M$. For $0 < t \leq \delta$ we have

$$\begin{aligned} &|x(t) - y(t)| \\ &\leq \int_0^t |f(s,x(s)) - f(s,y(s))|\, ds \\ &\leq 2\,\varepsilon \int_0^t u'(s)\, ds \leq 2\,\varepsilon\, u(t) \end{aligned}$$

so that $|x(t) - y(t)| = o(u(t))$ as $t \to 0^+$. Since

$$\begin{aligned} &|x(t) - y(t)| \\ &\leq \int_0^t |f(s,x(s)) - f(s,y(s))|\, ds \\ &\leq \int_0^t \frac{u'(s)}{u(s)}\, \omega(|x(s) - y(s)|)\, ds \end{aligned}$$

Lemma 2.3 yields $|x(t) - y(t)| \equiv 0$. □

3. Convergence of the successive approximations

The successive approximations for the the problem (1.1)-(1.2) are defined by the sequence of functions

$$x_i(t) = \int_0^t f(s, x_{i-1}(s))\, ds, \;\; i \geq 1, \tag{3.1}$$

$x_0(t)$ being a continuous function on $[0,a]$ such that $x_0(0) = 0$ and $|x_0(t)| \leq M$ for $0 \leq t \leq a$. It turns out that the hypotheses (1.4) and (1.5) guarantee not only uniqueness but also the convergence of the successive approximations.

Theorem 3.1. *If the hypotheses of Theorem 2.4 are satisfied, then there exists a sufficiently small interval $0 \leq t \leq c$, $c > 0$, on which the successive approximations exist and converge uniformly to the unique solution of (1.1)-(1.2).*

Proof. We first prove that the successive approximations $\{x_i(t)\}$ are well defined. From (1.4) it follows that, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) \in (0,a]$ such that

$$|f(s,x)| \leq \frac{\varepsilon}{2}\, u'(s), \;\; 0 < s \leq \delta, \;\; |x| \leq M.$$

Then it follows by (1.4) that for $t \in [0,\delta]$

$$\begin{aligned} |x_1(t)| &\leq \int_0^t |f(s,x_0(s))|\, ds \\ &\leq \frac{\varepsilon}{2} \int_0^t u'(s)\, ds \leq \frac{\varepsilon}{2}\, u(a). \end{aligned}$$

Taking

$$\varepsilon := \frac{2M}{u(a)}$$

we obtain

$$|x_1(t)| \le M \quad \text{for} \quad 0 \le t \le \delta.$$

Suppose now that for $j \ge 1$ the continuous function $x_{j-1}(t)$ is well defined on $[0,\delta]$ and satisfies $x_{j-1}(0) = 0$. We then see that $f(t, x_{j-1}(t))$ is well defined, continuous and the integral in (3.1) exists, and its norm does not exceed $\dfrac{\varepsilon}{2}\,u(a)$. This implies that $x_j(t)$ is also continuous and satisfies

$$x_j(0) = 0, \;\; |x_j(t)| \le M \;\; \text{for} \;\; 0 \le t \le \delta.$$

It follows that that the successive approximations are well defined and are uniformly bounded on $[0,\delta]$.

Now we prove that the family $\{x_j(t)\}$ is equicontinuous. Let $0 \le t_1 < t_2 \le \delta$ and $j \ge 1$ be given. Then

$$\begin{aligned} |x_j(t_2) - x_j(t_1)| &= \Big| \int_{t_1}^{t_2} f(s, x_{j-1}(s))\, ds \Big| \\ &\le \int_{t_1}^{t_2} \varepsilon\, u'(s)\, ds = \varepsilon\, [u(t_2) - u(t_1)]. \end{aligned}$$

From this and the first calculations it follows that $\{x_j(t)\}$ is equicontinuous and uniformly bounded on $[0,\delta]$. Then by the Arzela-Ascoli theorem, there exists a subsequence $\{x_{j_k}(t)\}$ which converges uniformly on $[0,\delta]$ to a continuous function $g(t)$ as $j_k \to \infty$. Since

$$x_{j_k+1}(t) = \int_0^t f(s, x_{j_k}(s))\, ds,$$

by continuity of f, the sequence $\{x_{j_k+1}(t)\}$ converges uniformly to

$$\tilde{g}(t) = \int_0^t f(s, g(s))\, ds.$$

We shall prove that on $[0,\delta]$ we have

$$\lim_{j\to\infty} |x_{j+1}(t) - x_j(t)| = 0. \tag{3.2}$$

By (3.1) this yields $g(t) = \tilde{g}(t)$ on $[0,\delta]$. This means that $g(t)$ is a solution of the equation. Since this solution is unique by Theorem 2.4, every subsequence of $\{x_j(t)\}$ which is convergent will tend to the same solution $g(t)$, and this shows that $\{x_j(t)\}$ converges to $g(t)$ on $[0,\delta]$. Because of the uniform boundedness and the equicontinuity of the sequence this convergence is uniform.

To prove (3.2) we define on $[0,\delta]$ the functions

$$\begin{aligned} y_j(t) &:= |x_{j+1}(t) - x_j(t)|\,, \quad j \ge 1, \\ m(t) &:= \sup_{0\le s\le t} \frac{|x_2(s) - x_1(s)|}{u(s)}, \\ z_1(t) &:= m(t)u(t). \end{aligned}$$

Then for $t \in [0,\delta]$ we have

$$0 \le m(t) \le \varepsilon$$

so that

$$0 \le z_1(t) \le \varepsilon\, u(t).$$

Also

$$\begin{aligned} y_j(t) &= |x_{j+1}(t) - x_j(t)| \\ &\le \int_0^t |f(s, x_j(s)) - f(s, x_{j-1}(s)))|\, ds \\ &\le \varepsilon \int_0^t u'(s)\, ds \le \varepsilon u(t), \end{aligned}$$

while

$$\begin{aligned} y_1(t) &\le \sup\nolimits_{0\le s\le t} \left\{ \frac{|x_2(s) - x_1(s)|}{u(s)} \right\} u(t) \\ &= m(t)u(t) = z_1(t). \end{aligned}$$

Define now on $[0, \delta]$ the functions z_j with $j \ge 1$ as follows:

$$z_{j+1}(t) := \int_0^t \frac{u'(s)}{u(s)}\, \omega(z_j(s))\, ds.$$

Since $0 \le z_1(t) \le \varepsilon u(t)$ and $u' \in L^1[0, a]$, the function z_2 is continuous on $[0, \delta]$ with

$$\begin{aligned} 0 \le z_2(t) &\le \int_0^t \frac{u'(s)}{u(s)}\, \omega(\varepsilon u(s))\, ds \\ &= \int_{\varepsilon u(0)}^{\varepsilon u(t)} \frac{\omega(r)}{r}\, dr \le \varepsilon\, u(t). \end{aligned}$$

By induction we show that for $j \ge 1$

$$0 \le z_j(t) \le \varepsilon\, u(t), \quad t \in [0, \delta]. \tag{3.3}$$

On the other hand,

$$\begin{aligned} y_2(t) &= |x_3(t) - x_2(t)| \\ &\le \int_0^t |f(s, x_2(s)) - f(s, x_1(s))|\ ds \\ &\le \int_0^t \frac{u'(s)}{u(s)}\, \omega(y_1(s)) \\ &\le \int_0^t \frac{u'(s)}{u(s)}\, \omega(z_1(s)) = z_2(t), \end{aligned}$$

and by induction one gets for $j \ge 1$ and $t \in [0, \delta]$ that

$$y_j(t) = |x_{j+1}(t) - x_j(t)| \le z_j(t). \tag{3.4}$$

We now prove by induction that for $j \ge 1$ and $t \in [0, \delta]$ we have

$$0 \le z_{j+1}(t) \le z_j(t). \tag{3.5}$$

Indeed,

$$\begin{aligned} & z_1(t) - z_2(t) \\ & = z_1(t) - \int_0^t \frac{u'(s)}{u(s)}\,\omega(z_1(s))\,ds \\ & = z_1(t) - \int_0^t \frac{u'(s)}{u(s)}\,\omega(m(s)u(s))\,ds \\ & \geq z_1(t) - \int_0^t \frac{u'(s)}{u(s)}\,\omega(m(t)u(s))\,ds \\ & = z_1(t) - \int_{m(t)u(0)}^{m(t)u(t)} \frac{\omega(r)}{r}\,dr \\ & \geq z_1(t) - \int_0^{m(t)u(t)} \frac{\omega(r)}{r}\,dr \\ & \geq z_1(t) - z_1(t) = 0. \end{aligned}$$

Now assume

$$z_j(t) \leq z_{j-1}(t), \qquad t \in [0,\delta].$$

Then

$$\begin{aligned} z_{j+1}(t) &= \int_0^t \frac{u'(s)}{u(s)}\,\omega(z_j(s))\,ds \\ &\leq \int_0^t \frac{u'(s)}{u(s)}\,\omega(z_{j-1}(s))\,ds = z_j(t) \end{aligned}$$

throughout $[0,\delta]$.

From (3.5) we infer that on $[0,\delta]$ the sequence $\{z_j(t)\}$ is decreasing and has a limit $z(t) \geq 0$ as $j \to \infty$. By Lebesgue's dominated convergence theorem we get

$$\begin{aligned} z(t) &= \lim_{j\to\infty} z_{j+1}(t) \\ &= \lim_{j\to\infty} \int_0^t \frac{u'(s)}{u(s)}\,\omega(z_j(s))\,ds \\ &= \int_0^t \lim_{j\to\infty} \Big\{ \frac{u'(s)}{u(s)}\,\omega(z_j(s)) \Big\}\,ds \\ &= \int_0^t \frac{u'(s)}{u(s)}\,\omega(\lim_{j\to\infty} z_j(s))\,ds \\ &= \int_0^t \frac{u'(s)}{u(s)}\,\omega(z(s))\,ds. \end{aligned}$$

Since $z(t) = o(u(t))$ for $t \downarrow 0$ cf. (3.3), by Lemma 2.1 it follows that $z \equiv 0$. From this and (3.4) we deduce (3.2) and the proof is complete. □

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